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α- LEFT JORDAN DERIVATIONS ON SOME BANACH ALGEBRAS

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Abstract

Let $\mathcal{A}$ be an algebra and $\mathcal{M}$ be a left $\mathcal{A}$- module. We say that a linear mapping $\delta : \mathcal{A} \to \mathcal{M}$ is an α- left derivation, if $\delta(AB) = \delta(A)\alpha(B) + \alpha(A)\delta(B)$ for any $A, B \in \mathcal{A}$. In this paper, we show that under certain conditions α- left Jordan derivations on some Banach algebras are zero.

1. Introduction

Let H denotes a complex separable Hilbert space. Let X be a complex Banach space and let $B(X)$ be the set of all bounded linear maps from X into itself.

A subspace lattice on X is a collection $\mathcal{L}$ of closed subspaces of X with (0), X in $\mathcal{L}$ and such that for every family $\{M_r\}$ of elements of $\mathcal{L}$, both $\bigcap M_r$ and $\vee M_r$ belong to $\mathcal{L}$, where $\vee M_r$ denotes the closed linear span of $\{M_r\}$. For a subspace lattice $\mathcal{L}$, alg $\mathcal{L}$ denotes the algebra of all operators on X that leave invariant each element of $\mathcal{L}$.

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A subspace lattice $\mathcal{L}$ on H is called a *commutative subspace lattice* (CSL), if it consists of mutually commuting projections and alg $\mathcal{L}$ is called a CSL algebra. A totally ordered subspace lattice $\mathcal{N}$ is called a *nest* and the associated algebra alg $\mathcal{N}$ is called a *nest algebra*. If $\mathcal{L}$ is a completely distributive commutative subspace lattice (CDCSL), then alg $\mathcal{L}$ is called a CDCSL algebra. It is obvious that a nest algebra is a CDCSL algebra. Given a subspace lattice $\mathcal{L}$ on X, put

$$\mathcal{J}_{\mathcal{L}} = \{K \in \mathcal{L} : K \neq \{0\} \text{ and } K_{-} \neq X\},$$

where $K_{-} = \vee\{L \in \mathcal{L} : K_{-} \subsetneq L\}$. Call $\mathcal{L}$ a $\mathcal{J}$-subspace lattice on X, if it satisfies the following conditions:

(i) $\vee\{a : a \in \mathcal{J}(L)\} = 1$,

(ii) $\wedge\{a_{-} : a \in \mathcal{J}(L)\} = 0$,

(iii) $a \wedge a_{-} = 0$ for every $a \in \mathcal{J}(L)$,

(iv) $a \vee a_{-} = 1$ for every $a \in \mathcal{J}(L)$.

If $\mathcal{L}$ is a $\mathcal{J}$-subspace lattice, then alg $\mathcal{L}$ is called a $\mathcal{J}$-SL algebra. For $x \in X$ and $f \in X^{*}$, the operator $y \to f(y)x$ is denoted by $(x \otimes f)y = f(y)x$. $\mathcal{F}(\mathcal{L})$ stands for the algebra of all finite rank operators in alg $\mathcal{L}$.

For notation, we use lower case letters to represent elements of rings and algebras in the abstract setting, and capital letters to represent elements of subalgebras of Hilbert space operators.

Let α be a surjective homomorphism on $\mathcal{A}$. A linear mapping δ from an algebra $\mathcal{A}$ to a left module $\mathcal{M}$ is called an α-*left Jordan derivation* for all $A \in \mathcal{A}$, $\delta(A^2) = 2\delta(A)\alpha(A)$.

In Section 2, we prove that for a right separating set I of $\mathcal{M}$, where $\mathcal{M}$ is a left $\mathcal{A}$-module and I is contained in the subalgebra of $\mathcal{A}$ generated by its idempotents, and for a surjective homomorphism α, if δ is an α-left Jordan derivation from $\mathcal{A}$ into $\mathcal{M}$, then $\delta \equiv 0$.

In Section 3, we give a result concern with a functional equation.

The following lemmas will be used repeatedly.

Lemma 1.1 [6, Lemma 3.1]. *Let $\mathcal{L}$ be a $\mathcal{J}$-subspace lattice on X. Then, the rank one operator $x \otimes f \in \operatorname{alg}\mathcal{L}$, if and only if there exists a subspace $K \in \mathcal{J}(\mathcal{L})$ such that $x \in K$ and $f \in K_{-}^{\perp}$.*

Lemma 1.2 [3, Lemma 2.10]. *Suppose that $\mathcal{L}$ is a $\mathcal{J}$-subspace lattice on X. Then, every rank one operator in* alg $\mathcal{L}$ *is contained in the linear span of the idempotents in $\mathcal{F}(\mathcal{L})$.*

2. α - Left Jordan Derivations

In this section, we assume that $\mathcal{A}$ is a unital algebra and $\mathcal{M}$ is any unital left $\mathcal{A}$-module.

Lemma 2.1. *Let α be a surjective homomorphism on $\mathcal{A}$. Let $\delta : \mathcal{A} \to \mathcal{M}$ be an α-Jordan left derivation. Then,*

(i) $\delta(AB + BA) = 2\alpha(A)\delta(B) + 2\alpha(B)\delta(A);$

(ii) $\delta(ABA) = \alpha(A^2)\delta(B) + 3\alpha(AB)\delta(A) - \alpha(BA)\delta(A).$

Proof. For any A, B in $\mathcal{A}$,

(i) $\delta(AB + BA) = \delta(AB) + \delta(BA)$

$$= \alpha(A)\delta(B) + \alpha(B)\delta(A) + \alpha(B)\delta(A) + \alpha(A)\delta(B)$$

$$= 2\alpha(A)\delta(B) + 2\alpha(B)\delta(A).$$

(ii) $\delta(ABA) = \frac{1}{2}[\delta(A(AB + BA) + (AB + BA)A) - \delta(A^2B + BA^2)]$

$$= \frac{1}{2}[2\alpha(A)\delta(AB + BA) + 2\alpha(AB + BA)\delta(A)$$

$$- 2\alpha(A^2)\delta(B) - 2\alpha(B)\delta(A^2)]$$

$$= \frac{1}{2}[2\alpha(A)(2\alpha(A)\delta(B) + 2\alpha(B)\delta(A)) + 2\alpha(AB)\delta(A)$$

$$+ 2\alpha(BA)\delta(A) - 2\alpha(A^2)\delta(B) - 4\alpha(B)\alpha(A)\delta(A^2)]$$

$$= \frac{1}{2}[4\alpha(A^2)\delta(B) + 4\alpha(AB)\delta(A) + 2\alpha(AB)\delta(A)$$

$$- 2\alpha(A^2)\delta(B) - 4\alpha(BA)\delta(A^2)]$$

$$= \frac{1}{2}[2\alpha(A^2)\delta(B) + 6\alpha(AB)\delta(A) - 2\alpha(BA)\delta(A)]$$

$$= \alpha(A^2)\delta(B) + 3\alpha(AB)\delta(A) - \alpha(BA)\delta(A).$$

Lemma 2.2. *Let α be a surjective homomorphism on $\mathcal{A}$. Let $\delta : \mathcal{A} \to \mathcal{M}$ be an α-left Jordan derivation. Then, for any $A \in \mathcal{A}$ and any idempotent $P \in \mathcal{A}$,*

(i) $\delta(P) = 0$;

(ii) $\delta(PA) = \delta(AP) = \alpha(P)\delta(A)$.

Proof. (i) For any idempotent P in $\mathcal{A}$, $\delta(P) = \delta(P^2) = 2\alpha(P)\delta(P)$. So, $\alpha(P)\delta(P) = 2\alpha(P^2)\delta(P) = 2\alpha(P)\delta(P)$. We have that $\alpha(P)\delta(P) = 0$. Thus

$$\delta(P) = 2\alpha(P)\delta(P) = 0. \tag{1}$$

(ii) By Lemma 2.1 and (1), for any $A \in \mathcal{A}$, $P = P^2 \in \mathcal{A}$,

$$\delta(AP + PAP) = \delta(APP + PAP) = 2\alpha(P)\delta(AP),$$

$$\delta(AP + PAP) = \delta(AP) + \delta(PAP) = \delta(AP) + \alpha(P)\delta(A).$$

So

$$2\alpha(P)\delta(AP) = \alpha(P)\delta(A) + \delta(AP).$$

Thus,

$$\alpha(P)\delta(AP) = \alpha(P)\delta(A).$$

We have that

$$\delta(AP) = \alpha(P)\delta(A). \tag{2}$$

Since $\delta(AP + PA) = 2\alpha(A)\delta(P) + 2\alpha(P)\delta(A) = 2\alpha(P)\delta(A)$, by (2),

$$\delta(PA) = 2\alpha(P)\delta(A) - \delta(AP) = \alpha(P)\delta(A). \tag{3}$$

By (2) and (3), $\delta(AP) = \delta(PA) = \alpha(P)\delta(A)$. By induction, it is easy to show the following result.

Lemma 2.3. *Let* α *be a surjective homomorphism on* $\mathcal{A}$. *If* δ *is an* α *- left Jordan derivation from* $\mathcal{A}$ *into* $\mathcal{M}$, *then for any idempotents* P_1, $P_2, \ldots, P_n$ *in* $\mathcal{A}$ *and any A in* $\mathcal{A}$,

$$\delta(P_1 \ldots P_n A) = \delta(AP_1 \ldots P_n) = \alpha(P_1 \ldots P_n)\delta(A) = \alpha(P_1)\ldots\alpha(P_n)\delta(A).$$

We call a right ideal $\mathcal{I}$ of $\mathcal{A}$ a right separating set of $\mathcal{M}$, if for any *m* in $\mathcal{M}$, $\mathcal{I}m = 0$ implies $m = 0$.

Theorem 2.4. *Let* $\mathcal{I}$ *be a right separating set of* $\mathcal{M}$. *Suppose that* $\mathcal{I}$ *is contained in the subalgebra of* $\mathcal{A}$ *generated by its idempotents. Let* α *be a surjective homomorphism on* $\mathcal{A}$. *If* δ *is an* α *- left Jordan derivation from* $\mathcal{A}$ *into* $\mathcal{M}$, *then* $\delta \equiv 0$. *In particular, if* δ *is an* α *- left derivation from* $\mathcal{A}$ *into* $\mathcal{M}$, *then* $\delta \equiv 0$.

Proof. By Lemma 2.3, for any $S \in \mathcal{I}$ and any $A \in \mathcal{A}$,

$$\delta(AS) = \delta(SA) = \alpha(S)\delta(A). \tag{4}$$

Since $\mathcal{I}$ is a right ideal, $TA \in \mathcal{I}$ for any $T \in \mathcal{I}$, $A \in \mathcal{A}$. Thus, for any $A \in \mathcal{A}$, $T \in \mathcal{I}$, by Lemma 2.2(i) and (4),

$$\alpha(T)\delta(A) = \delta(TA) = \alpha(TA)\delta(I) = 0. \tag{5}$$

Since $\mathcal{I}$ is a right separating set, it follows from (5) that, $\delta(A) = 0$ for any $A \in \mathcal{A}$.

Let $\mathcal{A}$ be an ultraweakly closed subalgebra of $B(H)$. The Banach space $\mathcal{M}$ is said to be a *dual normal Banach left* $\mathcal{A}$-*module*, if $\mathcal{M}$ is a Banach left $\mathcal{A}$-module, $\mathcal{M}$ is a dual space, and for any $m \in \mathcal{M}$, the map $\mathcal{A} \ni a \to am$ is ultraweak to *weak** continuous.

Corollary 2.5. *If* $\mathcal{L}$ *is a CDCSL on H. Let* α *be a surjective homomorphism on* $\text{alg}\,\mathcal{L}$ *and* δ *is an* α-*left Jordan derivation from* $\text{alg}\,\mathcal{L}$ *into a dual normal unital Banach left* $\text{alg}\,\mathcal{L}$-*module* $\mathcal{M}$, *then* $\delta \equiv 0$. *In particular, every* α-*left Jordan derivation from* $\text{alg}\,\mathcal{L}$ *into itself is equal to zero.*

Proof. Let $\mathcal{I} = \text{span}\{T : T \in \text{alg}\,\mathcal{L}, \text{rank}\, T = 1\}$. Then $\mathcal{I}$ is an ideal of $\text{alg}\,\mathcal{L}$. By [3, Lemma 2.3], $\mathcal{I}$ is contained in the linear span of the idempotents in $\text{alg}\,\mathcal{L}$. By [5, Theorem 3], we have that $\mathcal{I}$ is a right separating set $\mathcal{M}$. Hence, it follows from Theorem 2.4 that $\delta \equiv 0$.

Corollary 2.6. *Let* $\mathcal{L}$ *be a* $\mathcal{J}$-*subspace lattice on X. Let* α *be a surjective homomorphism on* $\text{alg}\,\mathcal{L}$. *If* δ *is an* α-*left Jordan derivation from* $\text{alg}\,\mathcal{L}$ *into itself, then* $\delta \equiv 0$.

Proof. Let $\mathcal{I} = \text{span}\{T : T \in \text{alg}\,\mathcal{L}, \text{rank}\, T = 1\}$. Then $\mathcal{I}$ is an ideal of $\text{alg}\,\mathcal{L}$. By Lemma 1.2, $\mathcal{I}$ is contained in the linear span of the idempotents in $\text{alg}\,\mathcal{L}$. By [5, Lemma 2.3], $\mathcal{I}$ is a right separating set of $\text{alg}\,\mathcal{L}$. Hence, it follows from Theorem 2.4 that $\delta \equiv 0$.

Corollary 2.7. *Suppose* $\mathcal{A}$ *is a unital Banach subalgebra of* $B(X)$ *such that* $\mathcal{A}$ *contains* $\{x_0 \otimes f, f \in X^*\}$, *where* $0 \neq x_0 \in X$. *Let* α *be a surjective homomorphism on* $\mathcal{A}$. *If* $\delta : \mathcal{A} \to B(X)$ *is an* α-*left Jordan derivation, then* $\delta \equiv 0$.

Proof. Let $\mathcal{I} = \{x_0 \otimes f, f \in X^*\}$. Then $\mathcal{I}$ is a right ideal of $\mathcal{A}$ and a right separating set of $B(X)$. For any $x_0 \otimes f$ in $\mathcal{A}$, if $f(x_0) \neq 0$, then

$\frac{1}{f(x_0)} x_0 \otimes f$ is an idempotent in $\mathcal{I}$. If $f(x_0) = 0$, choose $f_1(x_0) = 1$, we have that $x_0 \otimes f = \frac{1}{2} x_0 \otimes (f + f_1) - \frac{1}{2} x_0 \otimes (f_1 - f)$, both $x_0 \otimes (f + f_1)$ and $x_0 \otimes (f_1 - f)$ are idempotents. By Theorem 2.4, we have $\delta \equiv 0$.

Let $\mathcal{A}$ be a weakly closed subalgebra of $B(H)$. If K is a complex separable Hilbert space, then the tensor product $\mathcal{A} \otimes B(K)$ is defined as the weak operator closure of the span of all elementary tensors $A \otimes B$ acting on $H \otimes K$, where $A \in \mathcal{A}$ and $B \in B(H)$. A weakly closed subalgebra $\mathcal{A}$ of $B(H)$ is said to be *infinite multiplicity*, if $\mathcal{A} \otimes B(K)$ is isomorphic to $\mathcal{A}$.

Proposition 2.8. *Let $\mathcal{A}$ be a weakly closed unital subalgebra of $B(H)$ of infinite multiplicity. Let α be a surjective homomorphism on $\mathcal{A}$. If δ is an α-left Jordan derivation from $\mathcal{A}$ into a left $\mathcal{A}$-module $\mathcal{M}$, then $\delta \equiv 0$.*

Proof. By [8, Theorem 4.3], every $A \in \mathcal{A}$ is a sum of eight idempotents in $\mathcal{A}$. Thus, it follows from Lemma 2.2 that $\delta(A) = 0$.

Proposition 2.9. *Let $\mathcal{L}$ be a $\mathcal{J}$-subspace lattice on X. Let α be a surjective homomorphism on $\mathcal{F}(\mathcal{L})$. If δ is a linear mapping from $\mathcal{F}(\mathcal{L})$ into an algebra $\mathcal{B}$ such that $\delta(P) = 0$ for any idempotent $P \in \mathcal{F}(\mathcal{L})$, then $\delta \equiv 0$.*

Proof. For any $A, B \in \mathcal{F}(\mathcal{L})$, by [7, Proposition 3.2], we have that $A = A_1 + A_2 + \dots + A_n$, where $A_i = x_i \otimes f_i$ are rank one operators in $\operatorname{alg} \mathcal{L}$. It follows from Lemmas 1.1 and 2.2 that $\delta(A_i) = 0$, $i = 1, 2, \dots, n$. Thus, $\delta(A) = 0$ for any $A \in \mathcal{F}(\mathcal{L})$.

Corollary 2.10. *Let $\mathcal{L}$ be a $\mathcal{J}$-subspace lattice on X. Let α be a surjective homomorphism. If δ is an α-left Jordan derivation from $\mathcal{F}(\mathcal{L})$ into a left* $\operatorname{alg} \mathcal{L}$*-module $\mathcal{M}$, then $\delta \equiv 0$.*

Proposition 2.11. *Let* $\mathcal{L}$ *be a CSL on* H. *Let* α *be a surjective homomorphism on* $\operatorname{alg} \mathcal{L}$. *If* δ *is a bounded* α-*left Jordan derivation from* $\operatorname{alg} \mathcal{L}$ *into* $B(H)$, *then* $\delta \equiv 0$.

Proof. By Lemma 2.2(ii), for any $P = P^2 \in \operatorname{alg} \mathcal{L}$ and $A \in \operatorname{alg} \mathcal{L}$,

$$\delta(PA) = \delta(PPA) = \alpha(P)\delta(PA).$$

By [4, Theorem 2.20], $\delta(A) = \alpha(A)\delta(I)$, for any $A \in \operatorname{alg} \mathcal{L}$. It follows from Lemma 1.2(i) that $\delta(I) = 0$. Thus, $\delta(A) = 0$ for any $A \in \operatorname{alg} \mathcal{L}$.

Let $\mathcal{M}$ be a Banach left $\mathcal{A}$-module. A linear mapping D from $\mathcal{A}$ into $\mathcal{M}$ is an approximately local left derivation, if for each a in $\mathcal{A}$, there is a sequence of left derivations $\{D_{a,n}\}$ from $\mathcal{A}$ into $\mathcal{M}$ such that $\lim_{n\to\infty} D_{a,n}(a) = D(a)$. If in addition, D is bounded, then we say that D is a bounded approximately local derivation.

Let $\mathcal{A}$ be a Banach algebra and let $\mathcal{I}$ be the subalgebra of $\mathcal{A}$ generated by the idempotents in $\mathcal{A}$. We say that $\mathcal{A}$ is topologically generated by idempotents, if $\mathcal{I}$ is dense in $\mathcal{A}$.

Proposition 2.12. *Let* $\mathcal{A}$ *be a Banach algebra topologically generated by idempotents. Let* α *be a surjective homomorphism on* $\mathcal{A}$. *Then, every bounded approximately local* α-*left derivation from* $\mathcal{A}$ *into Banach left* $\mathcal{A}$-*module* $\mathcal{M}$ *is zero.*

Proof. For any idempotents $e_1, \ldots, e_m$ in $\mathcal{A}$, there is a sequence of α-left derivations $\{D_n\}$ from $\mathcal{A}$ into $\mathcal{M}$ such that $\lim_{n\to\infty} D_n(e_1 \ldots e_m) = D(e_1 \ldots e_m)$. Since every α-left derivation is α-left Jordan derivation, it follows from Lemma 2.2(i) and (5) that $D_n(e_1 \ldots e_m) = \alpha(e_1 \ldots e_{m-1})D_n(e_m) = 0$. Thus, $D(e_1 \ldots e_m) = 0$ for any idempotents $e_1, e_2, \ldots, e_m$ in $\mathcal{A}$. Since $\mathcal{A}$ is generated by idempotents and D is bounded, we have that $D \equiv 0$.

By the ideas in [2], we study the following functional equations by using Theorem 2.4.

Theorem 2.13. *Let $\mathcal{A}$ be a unital Banach algebra and $\mathcal{M}$ be a unital left $\mathcal{A}$ - module. Let α be a surjective homomorphism on $\mathcal{A}$. Suppose that $\mathcal{I}$ is a right separating set of $\mathcal{M}$ and $\mathcal{I}$ is contained in the subalgebra of $\mathcal{A}$ generated by idempotents. Let $f,\ g : \mathcal{A} \to \mathcal{M}$ be linear mappings. If*

$$f(A) = \alpha(A^2)g(A^{-1}), \tag{6}$$

holds for any invertible element A in $\mathcal{A}$, then the following statements hold:

(i) $f(A) = g(A)$ *for all* $A \in \mathcal{A}$;

(ii) $f(A) = \alpha(A)f(I)$ *for all* $A \in \mathcal{A}$.

Proof. (i) By (6), we have that

$$g(A) = \alpha(g^2)f(A^{-1}). \tag{7}$$

Let $D = f - g$. It follows from (6) and (7) that $D(A) = -\alpha(A^2)D\,(A^{-1})$ holds for any invertible element $A \in \mathcal{A}$. Then $D(I) = 0$. In the following, we prove that D is an α - left Jordan derivation. Since D is linear, we only need to show that

$$D(A^2) = 2\alpha(A)D(A), \tag{8}$$

for any $A \in \mathcal{A}$. Let $A \in \mathcal{A}$ be arbitrary. Choose an integer n such that B^{-1} and $(I - B)^{-1}$ exist, where $B = nI + A$. Thus, we have $B^2 = B - (B^{-1} + (I - B)^{-1})^{-1}$. Then,

$$\begin{aligned} D(B^2) &= D(B) - D((B^{-1} + (I - B)^{-1})^{-1}) \\ &= D(B) - \alpha(B^{-1} + (I - B)^{-1})^{-2} D(B^{-1} + (I - B)^{-1}) \\ &= D(B) - \alpha((I - B)^2 B^2 B^{-2})D(B) - \alpha(B^2 (I - B)^2 (I - B)^{-2})D(I - B) \\ &= D(B) - \alpha(I - B)^2 D(B) + \alpha(B^2)D(B) = 2\alpha(B)D(B). \end{aligned}$$

Hence $D(B^2) = 2\alpha(B)D(B)$, which implies (8) since $D(I) = 0$. Thus D is an α-left Jordan derivation from $\mathcal{A}$ into $\mathcal{M}$. By Theorem 2.4, it follows that $D \equiv 0$. Hence $f(A) = g(A)$ for any $A \in \mathcal{A}$. The relation (6) can be written in the form

$$f(A) = \alpha(A^2)f(A^{-1}). \tag{9}$$

(ii) Let us first assume that $f(I) = 0$. We want to show that $f = 0$.

For any $A \in \mathcal{A}$ and let us again choose an integer n such that B^{-1} and $(I - B)^{-1}$ exist, where $B = nI + A$. By (9), we have

$$\begin{aligned} f(B) &= \alpha(B^2)f(B^{-1}) = \alpha(B^2)f(B^{-1}(I - B)) \\ &= \alpha(B^2(B^{-1}(I - B))^2)f((I - B)^{-1}B) \\ &= \alpha((I - B)^2)f((I - B)^{-1} - I) \\ &= \alpha((I - B)^2((I - B)^{-1})^2)f(I - B) = -f(B). \end{aligned} \tag{10}$$

Hence $f(B) = 0$. Thus, $f(A) = 0$ for any $A \in \mathcal{A}$.

Now, we assume that $f(I) \neq 0$. Let $h(A) = f(A) - \alpha(A)f(I)$. It is obvious that h is linear. A routine calculation shows that $h(A) = \alpha(A^2)h(A^{-1})$ holds for any invertible operator $A \in \mathcal{A}$. Since $h(I) = 0$, we have $h(A) = 0$ for any $A \in \mathcal{A}$. Thus, $f(A) = \alpha(A)f(I)$ for any $A \in \mathcal{A}$.

Corollary 2.14. *Let $\mathcal{L}$ be a CDCSL or $\mathcal{J}$-subspace lattice on H and let $f, g : \text{alg } \mathcal{L} \to \text{alg } \mathcal{L}$ be linear mappings. Let α be a surjective homomorphism. Suppose that $f(A) = \alpha(A^2)g(A^{-1})$ holds for any invertible element A in $\mathcal{A}$. Then, the followings statements hold*:

(i) $f(A) = g(A)$ *for all* $A \in \text{alg } \mathcal{L}$;

(ii) $f(A) = \alpha(A)f(I)$ *for all* $A \in \text{alg } \mathcal{L}$.

Similar to the proof of Theorem 2.13, by Proposition 2.11, we can get the following theorem.

Theorem 2.15. *Let* $\mathcal{L}$ *be a CSL on H and let* $f, g : \text{alg } \mathcal{L} \to B(H)$ *be bounded linear mappings. Let* α *be a surjective homomorphism. Suppose that* $f(A) = \alpha(A^2)g(A^{-1})$ *holds for any invertible element A in* $\mathcal{A}$. *Then, the followings statements hold*:

(i) $f(A) = g(A)$ *for all* $A \in \text{alg } \mathcal{L}$;

(ii) $f(A) = \alpha(A)f(I)$ *for all* $A \in \text{alg } \mathcal{L}$.

3. Left Jordan Derivations

The following is a result concerning a functional equation.

Theorem 3.1. *Let* $\mathcal{A}$ *be a unital Banach algebra. Suppose that* $f : \mathcal{A} \to \mathcal{A}$ *is a linear mapping such that f is a left Jordan derivation on* $\mathcal{A}$. *Then,* $f(x) = -x^2 f(x^{-1})$ *for all invertible elements* $x \in \mathcal{A}$.

Proof. Suppose $x \in \mathcal{A}$, we have thus

$$\begin{aligned}\delta((x + x^{-1})^2) &= 2(x + x^{-1})\delta(x + x^{-1}) \\ &= 2x\delta(x) + 2x\delta(x^{-1}) + 2x^{-1}\delta(x) + 2x\delta(x^{-1}). \qquad (11)\end{aligned}$$

$$\begin{aligned}\delta((x + x^{-1})^2) &= \delta(x^2 + x^{-2} + 2xx^{-1}) \\ &= \delta(x^2) + \delta(x^{-2}) + 2\delta(xx^{-1}) \\ &= 2(x)\delta(x) + 2x^{-1}\delta(x^{-1}) + 2\delta(xx^{-1}). \qquad (12)\end{aligned}$$

From (11) and (12), $2\delta(xx^{-1}) = 2x^{-1}\delta(x) + 2x\delta(x^{-1})$. Since $\delta(I) = 0$, $0 = 2x^{-1}\delta(x) + 2x\delta(x^{-1})$, $0 = x^{-1}\delta(x) + x\delta(x^{-1})$. Thus $\delta(x) = -x^2\delta(x^{-1})$. This completes the proof.

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